

ON A GENERALIZED IDENTITY CONNECTING THETA SERIES ASSOCIATED WITH DISCRIMINANTS Δ AND Δp^2

FRANK PATANE

ABSTRACT. When the discriminants Δ and Δp^2 have one form per genus, [8] proves a theorem which connects the theta series associated to binary quadratic forms of each discriminant. This paper generalizes the main theorem of [8] by allowing Δ and Δp^2 to have multiple forms per genus. In particular, we state and prove an identity which connects the theta series associated to a single binary quadratic form of discriminant Δ to a theta series associated to a subset of binary quadratic forms of discriminant Δp^2 . Here and everywhere p is a prime.

1. INTRODUCTION

Let Δ be a discriminant of a positive definite binary quadratic form. When the discriminants Δ and Δp^2 have one form per genus, [8] gives an identity that connects the theta series associated to binary quadratic forms for each discriminant. This paper is mainly concerned with generalizing the central identity of [8] to discriminants which have multiple forms per genus. This generalized identity is stated in Theorem 5.1 where the discriminants Δ and Δp^2 are not required to have one form per genus. Theorem 5.1 gives an identity which connects a theta series associated to a binary quadratic form of discriminant Δ to a theta series associated to a subset of binary quadratic forms of discriminant Δp^2 .

Section 2 sets the notation and discusses some preliminary results. Section 3 considers a map of Buell which connects the class groups $\text{CL}(\Delta)$ and $\text{CL}(\Delta p^2)$. Section 4 contains the lemmas and identities which are necessary for the proof of Theorem 5.1. Section 5 combines the results of the previous sections to prove Theorem 5.1. Section 6 employs Theorem 5.1 to prove a general theorem given by [8, Theorem 5.1]. Lastly, Section 7 gives an explicit example which employs Theorem 5.1 to derive a Lambert series decomposition and the corresponding product representation formula.

2. PRELIMINARIES AND NOTATION

We use (a, b, c) to represent the class of binary quadratic forms which are equivalent to the binary quadratic form $ax^2 + bxy + cy^2$. Equivalence of two binary quadratic forms means the transformation matrix which connects them is in $SL(2, \mathbb{Z})$. The discriminant of (a, b, c) is defined as $\Delta := b^2 - 4ac$, and we only consider the case $\Delta < 0$ and $a > 0$. We say (a, b, c) is primitive when $\text{GCD}(a, b, c) = 1$. The set of all classes of primitive forms of discriminant Δ comprise what is known as the class group of discriminant Δ , denoted $\text{CL}(\Delta)$. We will often use the term “form” to mean a class of binary quadratic forms.

We use $h(\Delta) := |\text{CL}(\Delta)|$ to denote the class number of Δ . In the 1801 work, *Disquisitiones Arithmeticae*, Gauss develops much of the theory of binary quadratic forms, including the below relation between Δ and Δp^2 [5]:

$$(2.1) \quad h(\Delta p^2) = \frac{h(\Delta) \left(p - \left(\frac{\Delta}{p} \right) \right)}{w},$$

Date: May 23, 2016.

2010 *Mathematics Subject Classification.* 11E16, 11E25, 11F27, 11H55, 11R29.

Key words and phrases. binary quadratic forms, theta series, Lambert series.

where

$$(2.2) \quad w := \begin{cases} 3 & \Delta = -3, \\ 2 & \Delta = -4, \\ 1 & \Delta < -4. \end{cases}$$

The relation (2.1) as well as the number w in (2.2) appear in Section 2. Two binary quadratic forms of discriminant Δ are said to be in the same genus if they are equivalent over \mathbb{Q} via a transformation matrix in $SL(2, \mathbb{Q})$ whose entries have denominators coprime to 2Δ . An equivalent definition for the genera of binary quadratic forms is given by introducing the concept of assigned characters. The assigned characters of a discriminant Δ are the functions $\left(\frac{r}{p}\right)$ for all odd primes $p \mid \Delta$, as well as possibly the functions $\left(\frac{-1}{r}\right)$, $\left(\frac{2}{r}\right)$, and $\left(\frac{-2}{r}\right)$. The details are discussed in Buell [2] and in Cox [3].

The genera are of equal size and partition the class group. We say a discriminant is idoneal when each genus contains only one form. The number of genera of discriminant Δp^2 is either equal to the number of genera of discriminant Δ or double the number of genera of discriminant Δ . Letting $v(\Delta)$ be the number of genera of discriminant Δ we have

$$(2.3) \quad \frac{v(\Delta p^2)}{v(\Delta)} = \begin{cases} 1 & 2 < p, p \mid \Delta, \\ 2 & 2 < p, p \nmid \Delta, \\ 1 & p = 2, p \nmid \Delta, \\ 1 & p = 2, \Delta \equiv 0, 12, 28 \pmod{32}, \\ 2 & p = 2, \Delta \equiv 4, 8, 16, 20, 24 \pmod{32}. \end{cases}$$

The theta series associated to (a, b, c) is

$$(a, b, c, q) := \sum_{x, y} q^{ax^2 + bxy + cy^2} = \sum_{n \geq 0} (a, b, c; n) q^n,$$

where we use $(a, b, c; n)$ to denote the total number of representations of n by (a, b, c) . We define the projection operator $P_{m, r}$ to be

$$P_{m, r} \sum_{n \geq 0} a(n) q^n = \sum_{n \geq 0} a(mn + r) q^{mn+r},$$

where we take $0 \leq r < m$. Informally, the operator $P_{p, 0}$ applied to (a, b, c, q) collects the terms of (a, b, c, q) which have the exponent of q congruent to 0 \pmod{p} .

3. CONNECTING Δ TO Δp^2

Let (a, b, c) be a primitive form of discriminant Δ . In Chapter 7 of [2], Buell defines a map which sends (a, b, c) to a set of $p+1$ not necessarily distinct and not necessarily primitive forms of discriminant Δp^2 . The image of (a, b, c) under this map is given by

$$(3.1) \quad \{(a, bp, cp^2)\} \cup \{(ap^2, pb + 2ahp, ah^2 + bh + c) : 0 \leq h < p\}.$$

Buell devotes Section 7.1 of his book to determine the important properties of this map [2, Section 7.1]. Buell shows that if we cast out the nonprimitive forms of (3.1), then the remaining forms (all primitive) are repeated w times, where w is half the number of automorphs of Δ and is given in (2.2). We can map (a, b, c) to the set of distinct primitive forms of (3.1), and we call this set $\Psi_p(a, b, c)$.

Buell shows the images of Ψ_p are distinct, of the same size, and partition $\text{CL}(\Delta p^2)$ [2, Section 7.1]. Moreover, there are exactly $1 + \left(\frac{\Delta}{p}\right)$ nonprimitive forms in (3.1). Thus there are $p+1 - \left(1 + \left(\frac{\Delta}{p}\right)\right) =$

$p - \left(\frac{\Delta}{p}\right)$ primitive forms in (3.1). In other words, $|\Psi_p(a, b, c)| = \frac{p - \left(\frac{\Delta}{p}\right)}{w}$. Combining these results, Buell derives the class number of Δp^2 to be

$$(3.2) \quad h(\Delta p^2) = \frac{h(\Delta) \left(p - \left(\frac{\Delta}{p}\right)\right)}{w}.$$

TABLE 1. $\Delta = -3, p = 7$

h	Corresponding form of (3.1)
0	(1, 1, 37)
1	(3, 3, 13)
2	(7, 7, 7)
3	(3, 3, 13)
4	(7, 7, 7)
5	(3, 3, 13)
6	(1, 1, 37)

We emphasize that the only time there are repeated primitive forms in (3.1) is when $\Delta = -3, -4$. As an example we take $\Delta = -3$ and $p = 7$. The class group for $\Delta = -3$ consists of the single reduced form (1, 1, 1). The class group for discriminant $\Delta p^2 = -3 \cdot 7^2 = 147$ consists of the two reduced forms (1, 1, 37) and (3, 3, 13). The forms in (3.1) (counting repetition) consist of (1, 1, 37) union the forms listed in Table 1.

As expected, the primitive forms are repeated $w = 3$ times and there are $1 + (\frac{-3}{7}) = 2$ nonprimitive forms. We have $\Psi_7(1, 1, 1) = \{(1, 1, 37), (3, 3, 13)\}$. The preceding example illustrates the map Ψ_p when $w > 1$ and when (3.1) contains nonprimitive forms. If we apply Theorem 5.1 to this example we obtain identities which are discussed in [8].

As another example we let $\Delta = -55$ and $p = 3$. The genus structure along with the assigned characters for the genera for the discriminants $\Delta = -55$ and $\Delta p^2 = -495$ are given below.

	$\text{CL}(-55) \cong \mathbb{Z}_4$	$(\frac{r}{5})$	$(\frac{r}{11})$
g_1	(1, 1, 14), (4, 3, 4)	+1	+1
g_2	(2, 1, 7), (2, -1, 7)	-1	-1

	$\text{CL}(-495) \cong \mathbb{Z}_8 \times \mathbb{Z}_2$	$(\frac{r}{3})$	$(\frac{r}{5})$	$(\frac{r}{11})$
G_1	(1, 1, 124), (9, 9, 16), (4, 1, 31), (4, -1, 31),	+1	+1	+1
G_2	(5, 5, 26), (11, 11, 14), (9, 3, 14), (9, -3, 14),	-1	+1	+1
G_3	(2, 1, 62), (2, -1, 62), (8, 7, 17), (8, -7, 17),	-1	-1	-1
G_4	(7, 3, 18), (7, -3, 18), (10, 5, 13), (10, -5, 13),	+1	-1	-1

We compute

$$\begin{aligned}
 \Psi_3(1, 1, 14) &= \{(1, 1, 124), (9, 9, 16), (9, 3, 14), (9, -3, 14)\}, \\
 \Psi_3(4, 3, 4) &= \{(5, 5, 26), (11, 11, 14), (4, 1, 31), (4, -1, 31)\}, \\
 \Psi_3(2, 1, 7) &= \{(2, -1, 62), (7, -3, 18), (8, -7, 17), (10, 5, 13)\}, \\
 \Psi_3(2, -1, 7) &= \{(2, 1, 62), (7, 3, 18), (8, 7, 17), (10, -5, 13)\}.
 \end{aligned}
 \tag{3.3}$$

Also we see that

$$\begin{aligned}
 \Psi_3(g_1) &= G_1 \cup G_2, \\
 \Psi_3(g_2) &= G_3 \cup G_4.
 \end{aligned}
 \tag{3.4}$$

As expected, the images are distinct, of equal size, and partition $\text{CL}(\Delta p^2)$. Also we see in this example that $\Psi_p(f)$ is split evenly over two genera, and doesn't necessarily contain a form and its inverse. In general, the set $\Psi_p(f)$ will either be fully contained in one genus or be split equally between two genera. This behavior corresponds to whether $\frac{v(\Delta p^2)}{v(\Delta)} = 1, 2$, respectively. We refer the reader to (2.3) for the cases.

4. LEMMAS AND IDENTITIES

This section contains several lemmas and identities which we use to prove Theorem 5.1. Lemma 4.1 shows exactly which forms in (3.1) are nonprimitive.

Lemma 4.1. *Let (a, b, c) be a primitive form of discriminant Δ . There are exactly $1 + \left(\frac{\Delta}{p}\right)$ non-primitive forms in the list*

$$(4.1) \quad \{(a, bp, cp^2)\} \cup \{(ap^2, pb + 2ahp, ah^2 + bh + c) : 0 \leq h < p\},$$

and the nonprimitive forms are given by

$$(4.2) \quad \text{non-primitive forms of (4.1)} = \begin{cases} (a, bp, cp^2), & p \mid a, \left(\frac{\Delta}{p}\right) = 0, \\ f_1, & p \nmid a, \left(\frac{\Delta}{p}\right) = 0, \\ \emptyset, & \left(\frac{\Delta}{p}\right) = -1, \\ (a, pb, cp^2), f_2, & p \mid a, \left(\frac{\Delta}{p}\right) = 1, \\ f_3, f_4, & p \nmid a, \left(\frac{\Delta}{p}\right) = 1, \end{cases}$$

where $f_i := (ap^2, p(b + 2ah_i), ah_i^2 + bh_i + c)$. For p odd we take

$$(4.3) \quad h_1 \equiv \frac{-b}{2a} \pmod{p},$$

$$(4.4) \quad h_2 \equiv \frac{-c}{b} \pmod{p},$$

$$(4.5) \quad h_3 \equiv \frac{-b + \sqrt{\Delta}}{2a} \pmod{p},$$

$$(4.6) \quad h_4 \equiv \frac{-b - \sqrt{\Delta}}{2a} \pmod{p},$$

and for $p = 2$ we take $h_4 \not\equiv h_3 \equiv h_2 \equiv h_1 \equiv c \pmod{2}$. We always take $0 \leq h_i < p$.

Proof. Let (a, b, c) be a primitive form of discriminant $\Delta < 0$. Since (a, b, c) is assumed primitive, the only way for a form in (4.1) to be nonprimitive is if p divides each of its entries. Hence (a, bp, cp^2) is nonprimitive if and only if $p \mid a$. This takes care of the form (a, bp, cp^2) , and we are left with considering the forms $(ap^2, p(b + 2ah), ah^2 + bh + c)$, with $0 \leq h < p$.

The form $(ap^2, p(b + 2ah), ah^2 + bh + c)$ is nonprimitive if and only if $p \mid (ah^2 + bh + c)$, and so the remainder of the proof is devoted to determining exactly when $ah^2 + bh + c \equiv 0 \pmod{p}$.

Case $p \mid a$:

If $p \mid a$ and $p \mid b$, then $ah^2 + bh + c \equiv 0 \pmod{p}$ has no solutions since (a, b, c) is primitive. If $p \mid a$ and $p \nmid b$, then $ah^2 + bh + c \equiv 0 \pmod{p}$ has the unique solution $h \equiv \frac{-c}{b} \pmod{p}$, $0 \leq h < p$. We note that when $p \mid a$ and $p \nmid b$, we have $\Delta \equiv b^2 \pmod{p}$ and so $\left(\frac{\Delta}{p}\right) = 1$. We have found the form f_2 in (4.2).

Case $p \nmid a$, $p \neq 2$:

Since $p \nmid a$, we see

$$ah^2 + bh + c \equiv 0 \pmod{p}$$

is equivalent to

$$(2ah + b)^2 \equiv \Delta \pmod{p}.$$

We find the following forms are nonprimitive:

$$(4.7) \quad \begin{cases} f_1, & p \nmid a, \left(\frac{\Delta}{p}\right) = 0, \\ \emptyset, & p \nmid a, \left(\frac{\Delta}{p}\right) = -1, \\ f_3, f_4, & p \nmid a, \left(\frac{\Delta}{p}\right) = 1, \end{cases}$$

where $f_i := (ap^2, p(b + 2ah_i), ah_i^2 + bh_i + c)$ with

$$(4.8) \quad h_1 \equiv \frac{-b}{2a} \pmod{p},$$

$$(4.9) \quad h_3 \equiv \frac{-b + \sqrt{\Delta}}{2a} \pmod{p},$$

$$(4.10) \quad h_4 \equiv \frac{-b - \sqrt{\Delta}}{2a} \pmod{p},$$

and $0 \leq h_1, h_2, h_3 < p$.

Case $p \nmid a, p = 2$:

Since $2 \nmid a$, we have

$$ah^2 + bh + c \equiv h + bh + c \equiv (b+1)h + c \pmod{2}.$$

If $\left(\frac{\Delta}{2}\right) = 0$, then $2 \mid b$ and we have

$$(b+1)h + c \equiv 0 \pmod{2}$$

implies $h \equiv c \pmod{2}$. We have arrived at the nonprimitive form f_1 with $h_1 \equiv c \pmod{2}$.

If $\left(\frac{\Delta}{2}\right) = -1$ then $2 \nmid b$ and we have

$$\Delta \equiv 1 - 4ac \equiv 5 \pmod{8},$$

and so c is odd in this sub-case. Thus a, b, c are all odd and we see $(4a, 2b, c)$ and $(4a, 6b, a + b + c)$ are primitive. In other words, $(ap^2, pb + 2ahp, ah^2 + bh + c)$ with $h = 0, 1$ are both primitive forms. Hence we have only primitive forms in this sub-case.

If $\left(\frac{\Delta}{p}\right) = 1$ then $2 \nmid b$ and we have

$$\Delta \equiv 1 - 4ac \equiv 1 \pmod{8}$$

so that c is even. Thus a, b are odd and c is even implies both $(4a, 2b, c)$ and $(4a, 6b, a + b + c)$ are nonprimitive. Hence $(ap^2, pb + 2ahp, ah^2 + bh + c)$ with $h = 0, 1$ are both nonprimitive forms.

We now list the nonprimitive forms found in this case:

$$(4.11) \quad \begin{cases} f_1, & 2 \nmid a, \left(\frac{\Delta}{2}\right) = 0, \\ \emptyset, & 2 \nmid a, \left(\frac{\Delta}{2}\right) = -1, \\ f_3, f_4, & 2 \nmid a, \left(\frac{\Delta}{2}\right) = 1, \end{cases}$$

where $f_i := (ap^2, p(b + 2ah_i), ah_i^2 + bh_i + c)$ with

$$h_4 \not\equiv h_3 \equiv h_1 \equiv c \pmod{2},$$

and $0 \leq h_i < 2$.

We have considered all possible cases and completed the proof of Lemma 4.1. \square

Lemma 4.1 is essential to finding which forms are in $\Psi_p(a, b, c)$, and we are a step closer to proving Theorem 5.1. Before proving Theorem 5.1 we first consider $P_{p,0}(a, b, c, q)$ for an arbitrary primitive form (a, b, c) and prime p .

Lemma 4.2. *Let (a, b, c) be a primitive form of discriminant Δ . We have*

$$(4.12) \quad P_{p,0}(a, b, c, q) = \begin{cases} (a, bp, cp^2, q), & p \mid a, \left(\frac{\Delta}{p}\right) = 0, \\ f_1(q), & p \nmid a, \left(\frac{\Delta}{p}\right) = 0, \\ (a, b, c, q^{p^2}), & \left(\frac{\Delta}{p}\right) = -1, \\ f_2(q) + (a, pb, cp^2, q) - (a, b, c, q^{p^2}), & p \mid a, \left(\frac{\Delta}{p}\right) = 1, \\ f_3(q) + f_4(q) - (a, b, c, q^{p^2}), & p \nmid a, \left(\frac{\Delta}{p}\right) = 1, \end{cases}$$

where $f_i(q) := (ap^2, p(b + 2ah_i), ah_i^2 + bh_i + c, q)$. For p odd we take

$$(4.13) \quad h_1 \equiv \frac{-b}{2a} \pmod{p},$$

$$(4.14) \quad h_2 \equiv \frac{-c}{b} \pmod{p},$$

$$(4.15) \quad h_3 \equiv \frac{-b + \sqrt{\Delta}}{2a} \pmod{p},$$

$$(4.16) \quad h_4 \equiv \frac{-b - \sqrt{\Delta}}{2a} \pmod{p},$$

and for $p = 2$ we take $h_4 \not\equiv h_3 \equiv h_2 \equiv h_1 \equiv c \pmod{2}$. We always take $0 \leq h_i < p$.

Proof. The proof is split into cases.

Case $p \mid a$:

If $p \mid a$ and $p \mid \Delta$, then $p \mid b$ and $p \nmid c$ since (a, b, c) is assumed primitive. The congruence

$$ax^2 + bxy + cy^2 \equiv cy^2 \equiv 0 \pmod{p},$$

implies $y \equiv 0 \pmod{p}$, and we find $P_{p,0}(a, b, c, q) = (a, bp, cp^2, q)$.

If $p \mid a$, $p \nmid \Delta$ then $\Delta \equiv b^2 \not\equiv 0 \pmod{p}$ and so we must have $\left(\frac{\Delta}{p}\right) = 1$. Then

$$ax^2 + bxy + cy^2 \equiv y(bx + cy) \equiv 0 \pmod{p},$$

if and only if either $y \equiv 0 \pmod{p}$ or $x \equiv \frac{-c}{b}y \pmod{p}$. We have

$$\begin{aligned} P_{p,0} \sum_{x,y} q^{ax^2+bxy+cy^2} &= \sum_{\substack{x, \\ y \equiv 0 \pmod{p}}} q^{ax^2+bxy+cy^2} + \sum_{\substack{x \equiv \frac{-c}{b}y \pmod{p}, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2+bxy+cy^2} \\ &= (a, pb, cp^2, q) + \sum_{x \equiv \frac{-c}{b}y \pmod{p}} q^{ax^2+bxy+cy^2} - (a, b, c, q^{p^2}) \\ &= (a, pb, cp^2, q) + (ap^2, p(b + 2ah_2), ah_2^2 + bh_2 + c, q) - (a, b, c, q^{p^2}), \end{aligned}$$

where $h_2 \equiv \frac{-c}{b} \pmod{p}$ and Identity 4.5 is employed in the last equality.

Case $p \nmid a$, $p \neq 2$:

In this case, the congruence

$$ax^2 + bxy + cy^2 \equiv 0 \pmod{p},$$

is equivalent to

$$(4.17) \quad (2ax + by)^2 \equiv \Delta y^2 \pmod{p}.$$

If $\left(\frac{\Delta}{p}\right) = 0$ then (4.17) along with Identity 4.5 implies

$$P_{p,0}(a, b, c, q) = \sum_{x \equiv h_1 y \pmod{p}} q^{ax^2 + bxy + cy^2} = (ap^2, p(b + 2ah_1), ah_1^2 + bh_1 + c, q),$$

where $h_1 \equiv \frac{-b}{2a} \pmod{p}$.

If $\left(\frac{\Delta}{p}\right) = 1$ then (4.17) along with Identity 4.5 yields

$$P_{p,0}(a, b, c, q) = f_3(q) + f_4(q) - (a, b, c, q^{p^2}),$$

where

$$\begin{aligned} f_3(q) &= (ap^2, p(b + 2ah_3), ah_3^2 + bh_3 + c, q) \\ f_4(q) &= (ap^2, p(b + 2ah_4), ah_4^2 + bh_4 + c, q) \end{aligned}$$

and $h_3 \equiv \frac{-b + \sqrt{\Delta}}{2a} \pmod{p}$, $h_4 \equiv \frac{-b - \sqrt{\Delta}}{2a} \pmod{p}$.

Lastly we note that if $\left(\frac{\Delta}{p}\right) = -1$, then the only solutions to (4.17) is $x \equiv y \equiv 0 \pmod{p}$ and hence we have the theorem in this case. We have now finished the proof for p odd.

Case $p \nmid a$, $p = 2$:

If $\left(\frac{\Delta}{2}\right) = 0$ then $2 \mid b$ and we have

$$ax^2 + bxy + cy^2 \equiv x + cy \equiv 0 \pmod{2},$$

implies $x \equiv cy \pmod{2}$ is the only solution. Employing Identity 4.5 gives

$$P_{p,0}(a, b, c, q) = \sum_{x \equiv h_1 y \pmod{2}} q^{ax^2 + bxy + cy^2} = (ap^2, p(b + 2ah_1), ah_1^2 + bh_1 + c, q),$$

where $h_1 \equiv c \pmod{2}$.

If $\left(\frac{\Delta}{2}\right) = -1$ then $2 \nmid b$ and $\Delta \equiv 1 - 4ac \equiv 5 \pmod{8}$. Thus c is odd and we have

$$ax^2 + bxy + cy^2 \equiv x + xy + y \equiv 0 \pmod{2},$$

implies $x \equiv y \equiv 0 \pmod{2}$ is the only solution, and we have finished this case.

If $\left(\frac{\Delta}{2}\right) = 1$ then $2 \nmid b$ and $\Delta \equiv 1 - 4ac \equiv 1 \pmod{8}$. Thus c is even and we have

$$ax^2 + bxy + cy^2 \equiv x + xy \equiv 0 \pmod{2},$$

implies $x \equiv 0 \pmod{2}$ is a solution or $y \equiv 1 \pmod{2}$ is a solution. We find

$$\begin{aligned} P_{p,0}(a, b, c, q) &= (a, b, c, q^4) + \sum_{\substack{x \equiv 0 \pmod{2}, \\ y \not\equiv 0 \pmod{2}}} q^{ax^2 + bxy + cy^2} + \sum_{\substack{x \not\equiv 0 \pmod{2}, \\ y \not\equiv 0 \pmod{2}}} q^{ax^2 + bxy + cy^2} \\ &= (a, b, c, q^4) + \sum_{\substack{x, \\ y \not\equiv 0 \pmod{2}}} q^{ax^2 + bxy + cy^2}. \end{aligned}$$

Employing Identity 4.4 finishes the case, and hence the theorem. \square

We now state and prove some identities which will be of use in our proof of Theorem 5.1.

Identity 4.3. *Let (a, b, c) be a primitive form and $0 \leq h < p$. Then*

$$(4.18) \quad \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \equiv j \pmod{p}}} q^{ax^2 + (b+2ah)xy + (ah^2 + bh + c)y^2} = \sum_{\substack{x \equiv hj \pmod{p}, \\ y \equiv j \pmod{p}}} q^{ax^2 + bxy + cy^2}.$$

Proof. Use the change of variables $(x, y) \rightarrow (x - hy, y)$. \square

Identity 4.4. *Let (a, b, c) be a primitive form. Then*

$$(4.19) \quad \sum_{h=0}^{p-1} \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2 + (b+2ah)xy + (ah^2 + bh + c)y^2} = \sum_{\substack{x, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2 + bxy + cy^2}.$$

Proof. Sum (4.18) over $h = 0, 1, \dots, p-1$ and over $j = 1, 2, \dots, p-1$. Explicitly one gets

$$(4.20) \quad \sum_{h=0}^{p-1} \sum_{j=1}^{p-1} \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \equiv j \pmod{p}}} q^{ax^2 + (b+2ah)xy + (ah^2 + bh + c)y^2} = \sum_{h=0}^{p-1} \sum_{j=1}^{p-1} \sum_{\substack{x \equiv hj \pmod{p}, \\ y \equiv j \pmod{p}}} q^{ax^2 + bxy + cy^2}$$

$$(4.21) \quad = \sum_{\substack{x, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2 + bxy + cy^2}.$$

\square

Identity 4.5. *Let (a, b, c) be a primitive form and $0 \leq h < p$. Then*

$$(4.22) \quad \sum_{\substack{x \equiv 0 \pmod{p}, \\ y}} q^{ax^2 + (b+2ah)xy + (ah^2 + bh + c)y^2} = \sum_{x \equiv hy \pmod{p}} q^{ax^2 + bxy + cy^2}.$$

Proof. Sum (4.18) over $j = 0, 1, \dots, p-1$. Alternatively one may apply the change of variables $(x, y) \rightarrow (x - hy, y)$ directly. \square

Lemma 4.6. *Let $(A, B, C) \in CL(\Delta p^2)$. Then*

$$(4.23) \quad P_{p,0}(A, B, C, q) = (a, b, c, q^{p^2}),$$

where $(a, b, c) \in CL(\Delta)$ and $(A, B, C) \in \Psi_p(a, b, c)$.

Proof. By Section 3 we know $(A, B, C) \in CL(\Delta p^2)$ implies there exists a unique $(a, b, c) \in CL(\Delta)$ with $(A, B, C) \in \Psi_p(a, b, c)$. In other words, (A, B, C) is equivalent to either (a, bp, cp^2) or to $(ap^2, p(b + 2ah), ah^2 + bh + c)$ for some $0 \leq h < p$. Applying Lemma 4.2 along with Identity 4.5 completes the proof in both cases. \square

5. STATEMENT AND PROOF OF THEOREM 5.1

We have arrived at the main theorem of our paper.

Theorem 5.1. *Let (a, b, c) be a primitive form of discriminant $\Delta < 0$. For any prime p , we have*

$$(5.1) \quad w \sum_{(A, B, C) \in \Psi_p(a, b, c)} (A, B, C, q) = \left[p - \left(\frac{\Delta}{p} \right) \right] (a, b, c, q^{p^2}) + (a, b, c, q) - P_{p,0}(a, b, c, q).$$

We now prove Theorem 5.1. In all cases of the proof we start with the left hand side of (5.2)

$$(5.2) \quad w \sum_{(A,B,C) \in \Psi_p(a,b,c)} (A, B, C, q) - \left[p - \left(\frac{\Delta}{p} \right) \right] (a, b, c, q^{p^2}) = (a, b, c, q) - P_{p,0}(a, b, c, q),$$

and using the results of the previous sections, we end with the right hand side of (5.2). The proof is split according to the sign of $\left(\frac{\Delta}{p} \right)$ and if $p \mid a$. Throughout the proof we will always take $0 \leq h_i < p$.

Case $p \mid \Delta, p \mid a$:

By Lemma 4.1 we have $|\Psi_p(a, b, c)| = p$ and (a, bp, cp^2) is the only nonprimitive form listed in (3.1). Employing Identity 4.3 (with $j = 0$), Identity 4.4, and Lemma 4.2 we find that in this case we have

$$\begin{aligned} w \sum_{(A,B,C) \in \Psi_p(a,b,c)} (A, B, C, q) - \left[p - \left(\frac{\Delta}{p} \right) \right] (a, b, c, q^{p^2}) &= \sum_{h=0}^{p-1} \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2 + (b+2ah)xy + (ah^2 + bh + c)y^2} \\ &= \sum_{\substack{x, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2 + bxy + cy^2} \\ &= (a, b, c, q) - P_{p,0}(a, b, c, q), \end{aligned}$$

as desired.

Case $p \mid \Delta, p \nmid a$:

By Lemma 4.1, the only nonprimitive form in (3.1) is $(ap^2, p(b + 2ah_1), ah_1^2 + bh_1 + c, q)$, where for p odd we have $h_1 \equiv \frac{-b}{2a} \pmod{p}$ and for $p = 2$ we have $h_1 \equiv c \pmod{2}$. Employing Identity 4.3 (with $j = 0$), the left hand side of (5.2) is

$$(5.3) \quad \sum_{\substack{x \not\equiv 0 \pmod{p}, \\ y \equiv 0 \pmod{p}}} q^{ax^2 + bxy + cy^2} + \sum_{\substack{h=0, \\ h \not\equiv h_1 \pmod{p}}}^{p-1} \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2 + (b+2ah)xy + (ah^2 + bh + c)y^2}.$$

Employing Identity 4.4, we see (5.3) becomes

$$(5.4) \quad \sum_{\substack{x \not\equiv 0 \pmod{p}, \\ y \equiv 0 \pmod{p}}} q^{ax^2 + bxy + cy^2} + \sum_{\substack{x, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2 + bxy + cy^2} - \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2 + (b+2ah_1)xy + (ah_1^2 + bh_1 + c)y^2}.$$

Employing Identity 4.3 transforms (5.4) into

$$(5.5) \quad \sum_{\substack{x, \\ y \equiv 0 \pmod{p}}} q^{ax^2 + bxy + cy^2} + \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \equiv 0 \pmod{p}}} q^{ax^2 + bxy + cy^2} - (ap^2, p(b + 2ah_1), ah_1^2 + bh_1 + c, q).$$

It is clear that (5.5) is

$$(5.6) \quad (a, b, c, q) - P_{p,0}(a, b, c, q),$$

and we have finished this case.

Case $\left(\frac{\Delta}{p} \right) = -1$:

By Lemma 4.1 all forms of (3.1) are primitive in this case. Employing Identity 4.3 (with $j = 0$) the left hand side of (5.2) is

$$(5.7) \quad \sum_{\substack{x \not\equiv 0 \pmod{p}, \\ y \equiv 0 \pmod{p}}} q^{ax^2+bx+cy^2} + \sum_{h=0}^{p-1} \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2+(b+2ah)xy+(ah^2+bh+c)y^2}.$$

Employing Identity 4.4, we see (5.7) becomes

$$(5.8) \quad \sum_{\substack{x \not\equiv 0 \pmod{p}, \\ y \equiv 0 \pmod{p}}} q^{ax^2+bx+cy^2} + \sum_{\substack{x, \\ y \equiv 0 \pmod{p}}} q^{ax^2+bx+cy^2}.$$

Adding and subtracting (a, b, c, q^{p^2}) and using Lemma 4.2 gives that (5.8) is

$$(5.9) \quad (a, b, c, q) - P_{p,0}(a, b, c, q),$$

and we have finished this case.

Case $\left(\frac{\Delta}{p}\right) = 1, p \mid a$:

By Lemma 4.1 there are two nonprimitive forms in (3.1), and they are (a, pb, cp^2, q) , and $(ap^2, p(b+2ah_2), ah_2^2 + bh_2 + c, q)$, where $h_2 \equiv \frac{-c}{b} \pmod{p}$ (note this h_2 holds for $p = 2$ as well). Employing Identity 4.3 (with $j = 0$) we find the left hand side of (5.2) is

$$(5.10) \quad \sum_{h=0}^{p-1} \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2+(b+2ah)xy+(ah^2+bh+c)y^2} - \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2+(b+2ah_2)xy+(ah_2^2+bh_2+c)y^2}.$$

Employing Identity 4.4, we see (5.10) becomes

$$(5.11) \quad \sum_{\substack{x, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2+bx+cy^2} - \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2+(b+2ah_2)xy+(ah_2^2+bh_2+c)y^2}.$$

Adding and subtracting (a, b, c, q^{p^2}) and employing Identity 4.3 (with $j = 0$), we find (5.11) is

$$(5.12) \quad \sum_{\substack{x, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2+bx+cy^2} + (a, b, c, q^{p^2}) - \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2+(b+2ah_2)xy+(ah_2^2+bh_2+c)y^2}.$$

Lastly we add and subtract (a, bp, cp^2, q) in (5.12) to yield

$$(5.13) \quad (a, b, c, q) - (a, pb, cp^2, q) - (ap^2, p(b+2ah_2), ah_2^2 + bh_2 + c, q) + (a, b, c, q^{p^2}),$$

and applying Lemma 4.2 finishes this case.

Case $\left(\frac{\Delta}{p}\right) = 1, p \nmid a$:

By Lemma 4.1 there are two nonprimitive forms in (3.1), and they are $(ap^2, p(b+2ah_3), ah_3^2 + bh_3 + c, q)$, and $(ap^2, p(b+2ah_4), ah_4^2 + bh_4 + c, q)$, where for p odd we take $h_3 \equiv \frac{-b+\sqrt{\Delta}}{2a} \pmod{p}$, $h_4 \equiv \frac{-b-\sqrt{\Delta}}{2a} \pmod{p}$, and for $p = 2$ we can simply take $h_3 \not\equiv h_4 \pmod{2}$. Employing Identity 4.3 (with $j = 0$) along with Identity 4.4 shows the left hand side of (5.2) to be

$$(5.14) \quad \sum_{\substack{x \not\equiv 0 \pmod{p}, \\ y \equiv 0 \pmod{p}}} q^{ax^2+bx+cy^2} + \sum_{\substack{x, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2+bx+cy^2} - \sum_{i=3}^4 \sum_{\substack{x \equiv 0 \pmod{p}, \\ y \not\equiv 0 \pmod{p}}} q^{ax^2+(b+2ah_i)xy+(ah_i^2+bh_i+c)y^2}.$$

Adding and subtracting $2(a, b, c, q^{p^2})$ and employing Identity 4.3 (with $j = 0$), (5.14) becomes

$$(5.15) \quad (a, b, c, q) - (ap^2, p(b+2ah_3), ah_3^2 + bh_3 + c, q) - (ap^2, p(b+2ah_4), ah_4^2 + bh_4 + c, q) + (a, b, c, q^{p^2}).$$

Applying Lemma 4.2 finishes this case, and the theorem is proven.

6. RELATING THEOREM 5.1 TO [8, Theorem 5.1]

In this section we use Theorem 5.1 to prove Theorem 5.1 of [8]. First we give an example to illustrate the difference between Theorem 5.1 and [8, Theorem 5.1]. In Section 3 we discuss the map Ψ_3 between the class groups $\text{CL}(-55)$ and $\text{CL}(-55 \cdot 3^2)$. We continue this example by examining one of the identities of Theorem 5.1 with $\Delta = -55$ and $p = 3$. Theorem 5.1 yields

$$(6.1) \quad (1, 1, 124, q) + (9, 9, 16, q) + 2(9, 3, 14, q) = 4(1, 1, 14, q^9) + P_{3,1}(1, 1, 14, q) + P_{3,2}(1, 1, 14, q).$$

In general, Theorem 5.1 yields identities which are dissections modulo p of the theta series on the left hand side of (5.1). Equation (6.1) is a dissection modulo 3 of $(1, 1, 124, q) + (9, 9, 16, q) + 2(9, 3, 14, q)$. Furthermore we see the forms $(1, 1, 124)$, $(9, 9, 16)$ share a genus which is different than the genus containing $(9, 3, 14)$. See Section 3 for the genus structure of $\text{CL}(-55 \cdot 3^2)$. The forms $(1, 1, 124)$, $(9, 9, 16)$ are in a different genus than $(9, 3, 14)$ because they have different assigned character values for the character $(\frac{\cdot}{3})$. In other words, if $(1, 1, 124; n) + (9, 9, 16; n) > 0$ and $3 \nmid n$ then $n \equiv 1 \pmod{3}$. Similarly if $(9, 3, 14; n) > 0$ and $3 \nmid n$ then $n \equiv 2 \pmod{3}$. Employing Lemma 4.6 along with the above discussion allows us to separate (6.1) into the two identities

$$(6.2) \quad \begin{aligned} (1, 1, 124, q) + (9, 9, 16, q) &= 2(1, 1, 14, q^9) + P_{3,1}(1, 1, 14, q), \\ 2(9, 3, 14, q) &= 2(1, 1, 14, q^9) + P_{3,2}(1, 1, 14, q). \end{aligned}$$

Theorem 5.1 of [8] directly claims the identities of (6.2). Theorem 5.1 of [8] is Theorem 5.1 with the addition that we consider the congruence conditions implied by the assigned characters of the genera. An example is when the left hand side of (5.1) contains theta series associated to forms of two genera, we break (5.1) into two identities whose sum is (5.1). We now state [8, Theorem 5.1].

Theorem 6.1. *Let (a, b, c) be a primitive form of discriminant Δ , and G a genus of discriminant Δp^2 with $\Psi_{G,p}(a, b, c)$ nonempty. For p an odd prime, we have*

$$(6.3) \quad w \sum_{(A, B, C) \in \Psi_{G,p}(a, b, c)} (A, B, C, q) = w |\Psi_{G,p}(a, b, c)| (a, b, c, q^{p^2}) + \sum_{i=1}^{p-1} \frac{\left(\frac{ri}{p}\right)+1}{2} P_{p,i}(a, b, c, q)$$

and for $p = 2$,

$$(6.4) \quad w \sum_{(A, B, C) \in \Psi_{G,2}(a, b, c)} (A, B, C, q) = w |\Psi_{G,2}(a, b, c)| (a, b, c, q^4) + P_{2^{t+1},r}(a, b, c, q)$$

where

$$w := \begin{cases} 3 & \Delta = -3, \\ 2 & \Delta = -4, \\ 1 & \Delta < -4, \end{cases}$$

r is coprime to Δp^2 and is represented by any form of $\Psi_{G,p}(a, b, c)$. When $\Delta \equiv 0 \pmod{16}$ we define $t = 2$, and for $\Delta \not\equiv 0 \pmod{16}$ we define $t = 0, 1$ according to whether Δ is odd or even.

Here $\Psi_{G,p}(a, b, c) := \Psi_p(a, b, c) \cap G$, and all other notation is consistent with our notation. We note that the coefficient $\frac{\left(\frac{ri}{p}\right)+1}{2}$ of $P_{p,i}(a, b, c, q)$ is simply the integer 0 or 1 depending on the congruence class of ri .

Proof. Our proof naturally splits according to the parity of p and whether $p \mid \Delta$. In general, both (6.3) and (6.4) are dissections modulo p of the theta series

$$\sum_{(A,B,C) \in \Psi_{G,p}(a,b,c)} (A, B, C, q).$$

Lemma 4.6 shows

$$P_{p,0} \sum_{(A,B,C) \in \Psi_{G,p}(a,b,c)} (A, B, C, q) = |\Psi_{G,p}(a, b, c)|(a, b, c, q^{p^2}),$$

which is the 0 modulo p dissection of (6.3) and (6.4). Our proof now breaks into cases.

Case p odd, $p \nmid \Delta$:

Theorem 5.1 gives the identity

$$(6.5) \quad w \sum_{i=1}^{p-1} \sum_{(A,B,C) \in \Psi_p(a,b,c)} P_{p,i}(A, B, C, q) = \sum_{i=1}^{p-1} P_{p,i}(a, b, c, q).$$

In the case when p is odd and $p \nmid \Delta$ we know from Section 3 that $\Psi_p(a, b, c)$ is split equally over two genera which have the same assigned characters except for the character $\left(\frac{\bullet}{p}\right)$. Let G_1 be the genus with assigned character $\left(\frac{\bullet}{p}\right) = 1$, G_2 the genus with assigned character $\left(\frac{\bullet}{p}\right) = -1$, and $\Psi_p(a, b, c)$ is contained in $G_1 \cup G_2$. The left hand side of (6.5) is

$$(6.6) \quad w \sum_{i=1}^{p-1} \sum_{(A,B,C) \in \Psi_{G_1,p}(a,b,c)} P_{p,i}(A, B, C, q) + w \sum_{i=1}^{p-1} \sum_{(A,B,C) \in \Psi_{G_2,p}(a,b,c)} P_{p,i}(A, B, C, q).$$

The right hand side of (6.5) is

$$(6.7) \quad \sum_{\substack{i=1 \\ \left(\frac{i}{p}\right)=1}}^{p-1} P_{p,i}(a, b, c, q) + \sum_{\substack{i=1 \\ \left(\frac{i}{p}\right)=-1}}^{p-1} P_{p,i}(a, b, c, q).$$

If $(A, B, C) \in G_1$ and $(A, B, C; r) > 0$ for some r coprime to Δp^2 , then $(A, B, C; n) = 0$ for any n with $\left(\frac{n}{p}\right) = -1$. Similarly if $(A, B, C) \in G_2$ and $(A, B, C; r) > 0$ for some r coprime to Δp^2 , then $(A, B, C; n) = 0$ for any n with $\left(\frac{n}{p}\right) = 1$. We arrive at the identities

$$(6.8) \quad \begin{aligned} w \sum_{i=1}^{p-1} \sum_{(A,B,C) \in \Psi_{G_1,p}(a,b,c)} P_{p,i}(A, B, C, q) &= \sum_{\substack{i=1 \\ \left(\frac{i}{p}\right)=1}}^{p-1} P_{p,i}(a, b, c, q), \\ w \sum_{i=1}^{p-1} \sum_{(A,B,C) \in \Psi_{G_2,p}(a,b,c)} P_{p,i}(A, B, C, q) &= \sum_{\substack{i=1 \\ \left(\frac{i}{p}\right)=-1}}^{p-1} P_{p,i}(a, b, c, q), \end{aligned}$$

which shows Theorem 6.1 when p is odd and $p \nmid \Delta$.

TABLE 2

Δ	assigned characters
$\Delta \equiv 4 \pmod{16}$	χ_1, \dots, χ_r
$\Delta \equiv 12 \pmod{16}$	$\chi_1, \dots, \chi_r, \delta$
$\Delta \equiv 24 \pmod{32}$	$\chi_1, \dots, \chi_r, \delta\epsilon$
$\Delta \equiv 8 \pmod{32}$	$\chi_1, \dots, \chi_r, \epsilon$
$\Delta \equiv 16 \pmod{32}$	$\chi_1, \dots, \chi_r, \delta$
$\Delta \equiv 0 \pmod{32}$	$\chi_1, \dots, \chi_r, \delta, \epsilon$

Case p odd, $p \mid \Delta$:

We now show Theorem 6.1 when p is odd and $p \mid \Delta$. In this case, $\Psi_p(a, b, c) \subseteq G$. Since p is odd and $p \mid \Delta$, the character $\left(\frac{\bullet}{p}\right)$ is one of the assigned characters for the discriminant Δ . If $(a, b, c) \in \text{CL}(\Delta)$ is in a genus g which has $\left(\frac{\bullet}{p}\right) = 1$ then $P_{p,r}(a, b, c, q) = 0$ for any r with $\left(\frac{r}{p}\right) = -1$. In this case, showing (6.3) is equivalent to showing

$$(6.9) \quad w \sum_{(A,B,C) \in \Psi_{G,p}(a,b,c)} (A, B, C, q) = w |\Psi_p(a, b, c)|(a, b, c, q^{p^2}) + \sum_{i=1}^{p-1} P_{p,i}(a, b, c, q),$$

where we used $\Psi_{G,p}(a, b, c) = \Psi_p(a, b, c)$ and $P_{p,r}(a, b, c, q) = 0$ for any r with $\left(\frac{r}{p}\right) = -1$. Equation (6.9) is exactly (5.1). The case when $(a, b, c) \in \text{CL}(\Delta)$ is in a genus g with assigned character $\left(\frac{\bullet}{p}\right) = -1$ follows similarly.

Case $p = 2$, $p \nmid \Delta$:

When $p = 2$ and $p \nmid \Delta$ then (6.4) becomes

$$(6.10) \quad w \sum_{(A,B,C) \in \Psi_{G,2}(a,b,c)} (A, B, C, q) = w |\Psi_{G,2}(a, b, c)|(a, b, c, q^4) + P_{2,1}(a, b, c, q),$$

which is equivalent to (5.1).

Case $p = 2$, $p \mid \Delta$:

Lastly we have the case when $p = 2$ and Δ is even. Due to the nature of the assigned characters of a genus, there are several cases to consider when $p = 2$ and Δ is even. This is apparent from (2.3) as well as examining whether the characters $\delta := \left(\frac{-1}{r}\right)$, $\epsilon := \left(\frac{2}{r}\right)$, and $\delta\epsilon := \left(\frac{-2}{r}\right)$, are part of the assigned character list for Δ and 4Δ . Details regarding the congruence conditions when Δ contains the assigned characters δ , ϵ , and $\delta\epsilon$ are given in [2] and [3]. The assigned characters for even discriminants are given in Table 2, with $\chi_i := \left(\frac{\bullet}{p_i}\right)$ and p_i is an odd prime dividing Δ where i runs up to the number of distinct odd primes dividing Δ .

Our proof now splits according to whether $\frac{v(\Delta p^2)}{v(\Delta)} = 1, 2$ along with congruence conditions on Δ . In all of these cases we have $|\Psi_2(a, b, c)| = 2$ unless $\Delta = -4$. If $\Delta = -4$ then Theorem

6.1 directly reduces to Theorem 5.1 which reduces to the main theorem of [8] since both -4 and -16 are idoneal discriminants.

We first consider the case when $\frac{v(\Delta p^2)}{v(\Delta)} = 1$ which implies Ψ_2 maps into a single genus. Hence Theorem 6.1 reduces to Theorem 5.1 if can show

$$(6.11) \quad P_{2^{t+1},r}(a, b, c, q) = P_{2,1}(a, b, c, q),$$

where t is given in Theorem 6.1 and r is coprime to 2Δ and represented by (a, b, c) . Equation (2.3) implies that we need to consider $\Delta \equiv 0 \pmod{32}$, or $\Delta \equiv 12 \pmod{16}$. When $\Delta \equiv 0 \pmod{32}$, (6.11) becomes

$$(6.12) \quad P_{8,r}(a, b, c, q) = P_{2,1}(a, b, c, q).$$

Equation (6.12) is equivalent to showing $(a, b, c; s) = 0$ for all odd s coprime to Δ and $s \not\equiv r \pmod{8}$. This congruence condition follows from the fact that when $\Delta \equiv 0 \pmod{32}$, both Δ and 4Δ have the same assigned characters which are χ_p, δ, ϵ for all odd primes $p \mid \Delta$.

Similarly when $\Delta \equiv 12 \pmod{16}$ then (6.11) becomes

$$(6.13) \quad P_{4,r}(a, b, c, q) = P_{2,1}(a, b, c, q).$$

Equation (6.13) is equivalent to showing $(a, b, c; s) = 0$ for all odd s coprime to Δ and $s \not\equiv r \pmod{4}$. This congruence condition follows from the fact that when $\Delta \equiv 12 \pmod{16}$, both Δ and 4Δ have the same assigned characters which are χ_p, δ for all odd primes $p \mid \Delta$.

We are now left with the cases which all have $\frac{v(\Delta p^2)}{v(\Delta)} = 2$, and so Ψ_2 consists of two forms in different genera. In other words, we are left with the cases in which Δp^2 has exactly one additional character than Δ . Examining Table 2, we see these are the cases when $\Delta \equiv 4 \pmod{16}$ and $\Delta \equiv 8, 16, 24 \pmod{32}$. Let us call this one additional character λ . For example, when $\Delta \equiv 4 \pmod{16}$ the assigned characters of Δ are χ_1, \dots, χ_m and the assigned characters of 4Δ are $\chi_1, \dots, \chi_m, \delta$. In this example, λ would be the character δ . By taking λ to be a general character we can prove the remaining cases together.

Fix $(a, b, c) \in \text{CL}(\Delta)$. Let G_1 be the genus of 4Δ with assigned character $\lambda = 1$, G_2 the genus of 4Δ with assigned character $\lambda = -1$, and $\Psi_2(a, b, c) = \{(A, B, C), (D, E, F)\}$ so that $(A, B, C) \in G_1$ and $(D, E, F) \in G_2$. Theorem 5.1 gives

$$(6.14) \quad (A, B, C, q) + (D, E, F, q) = 2(a, b, c, q^4) + P_{2,1}(a, b, c, q).$$

We can write $P_{2,1}(a, b, c, q) = P_{2^k, r_1}(a, b, c, q) + P_{2^k, r_2}(a, b, c, q)$ where $\lambda(r_1) = 1$, $\lambda(r_2) = -1$, and k is 2 or 3 depending on the character λ . Employing Lemma 4.6 yields the identities

$$(6.15) \quad \begin{aligned} (A, B, C, q) &= (a, b, c, q^4) + P_{2^k, r_1}(a, b, c, q), \\ (D, E, F, q) &= (a, b, c, q^4) + P_{2^k, r_2}(a, b, c, q). \end{aligned}$$

The identities given in (6.15) are the identities of Theorem 6.1, and we have finished the proof of Theorem 6.1. □

7. LAMBERT SERIES AND PRODUCT REPRESENTATION FORMULAS

One of the main applications of Theorem 6.1 is that we are often able to deduce a Lambert series decomposition of the left hand side of (6.3) and (6.4), and hence a product representation formula for the associated forms. Theorem 6.1 yields a Lambert series decomposition only when the theta series on the left hand side of (6.3) and (6.4) are associated to the entire genus. We illustrate this property

with the example of $\Delta = -23$ and $p = 3$.

Let $\Delta = -23$ and $p = 3$. The class group and genus structure for the relevant discriminants is given by

$\text{CL}(-23) \cong \mathbb{Z}_3$	$\left(\frac{r}{23}\right)$
$(1, 1, 6), (2, 1, 3), (2, -1, 3)$	$+1$

$\text{CL}(-207) \cong \mathbb{Z}_6$	$\left(\frac{r}{23}\right)$	$\left(\frac{r}{3}\right)$
$(1, 1, 52), (4, 1, 13), (4, -1, 13)$	$+1$	$+1$
$(8, 7, 8), (2, 1, 26), (2, -1, 26)$	$+1$	-1

We compute

$$\begin{aligned}
 \Psi_3(1, 1, 6) &= \{(1, 1, 52), (8, 7, 8)\}, \\
 \Psi_3(2, 1, 3) &= \{(2, -1, 26), (4, 1, 13)\}, \\
 \Psi_3(2, -1, 3) &= \{(2, 1, 26), (4, -1, 13)\}.
 \end{aligned}
 \tag{7.1}$$

Employing Theorem 5.1 yields the identities

$$\begin{aligned}
 (1, 1, 52, q) + (8, 7, 8, q) &= 2(1, 1, 6, q^9) + (P_{3,1} + P_{3,2})(1, 1, 6, q), \\
 (2, 1, 26, q) + (4, 1, 13, q) &= 2(2, 1, 3, q^9) + (P_{3,1} + P_{3,2})(2, 1, 3, q).
 \end{aligned}
 \tag{7.2}$$

Either by Theorem 6.1 or by employing congruences directly to (7.2), we find

$$\begin{aligned}
 (1, 1, 52, q) &= (1, 1, 6, q^9) + P_{3,1}(1, 1, 6, q), \\
 (8, 7, 8, q) &= (1, 1, 6, q^9) + P_{3,2}(1, 1, 6, q), \\
 (4, 1, 13, q) &= (2, 1, 3, q^9) + P_{3,1}(2, 1, 3, q), \\
 (2, 1, 26, q) &= (2, 1, 3, q^9) + P_{3,2}(2, 1, 3, q).
 \end{aligned}
 \tag{7.3}$$

The identities of (7.2) and (7.3) do not directly yield Lambert series decompositions since the left hand sides are not associated with the entire genus. However we can combine the identities of (7.3) to find

$$\begin{aligned}
 (1, 1, 52, q) + 2(4, 1, 13, q) &= f(q^9) + P_{3,1}f(q), \\
 (8, 7, 8, q) + 2(2, 1, 26, q) &= f(q^9) + P_{3,2}f(q),
 \end{aligned}
 \tag{7.4}$$

where $f(q) = (1, 1, 6, q) + 2(2, 1, 3, q)$ is the theta series associated with the principal genus of Δ . The identities of (7.4) yield Lambert series decompositions and we demonstrate how to derive these Lambert series and product representation formulas.

Dirichlet's formula for quadratic forms gives $f(q)$ as a Lambert series

$$f(q) := (1, 1, 6, q) + 2(2, 1, 3, q) = 3 + 2 \sum_{n=1}^{\infty} \left(\frac{-23}{n}\right) \frac{q^n}{1-q^n}.
 \tag{7.5}$$

Using (7.5) it is not hard to show

$$(P_{3,1} - P_{3,2})f(q) = 2 \sum_{n=1}^{\infty} \left(\frac{69}{n}\right) \frac{q^n(1-q^n)}{1-q^{3n}}.
 \tag{7.6}$$

For convenience we define the Lambert series

$$L_1(q) = \sum_{n=1}^{\infty} \left(\frac{-23}{n}\right) \frac{q^n}{1-q^n},
 \tag{7.7}$$

and

$$(7.8) \quad L_2(q) = \sum_{n=1}^{\infty} \left(\frac{69}{n} \right) \frac{q^n(1-q^n)}{1-q^{3n}}.$$

It is easy to show

$$(7.9) \quad P_{3,0}L_1(q) = 2L_1(q^3) - L_1(q^9).$$

Adding and subtracting the identities of (7.4) and employing (7.6), (7.9), gives the Lambert series decompositions for the theta series associated with the genera of discriminant -207

$$(7.10) \quad (1, 1, 52, q) + 2(4, 1, 13, q) = 3 + L_1(q) - 2L_1(q^3) + 3L_1(q^9) + L_2(q),$$

and

$$(7.11) \quad (8, 7, 8, q) + 2(2, 1, 26, q) = 3 + L_1(q) - 2L_1(q^3) + 3L_1(q^9) - L_2(q).$$

Both (7.10) and (7.11) yield product representation formulas as we now demonstrate. We use the notation

$$[q^k] \sum_{n \geq 0} a(n)q^n = a(k),$$

so that $[q^k]f(q)$ is simply the coefficient of q^k in the expansion of the series $f(q)$. The coefficient of q^n in $L_1(q)$ is given by

$$A(n) := [q^n] \sum_{n=1}^{\infty} \left(\frac{-23}{n} \right) \frac{q^n}{1-q^n} = \sum_{d|n} \left(\frac{-23}{d} \right).$$

We see that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{69}{n} \right) \frac{q^n(1-q^n)}{1-q^{3n}} &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left(\frac{69}{n} \right) (q^{n(3m+1)} - q^{n(3m+2)}) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{69}{n} \right) (q^{n(3m-1)} - q^{n(3m-2)}) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{69}{n} \right) \left(\frac{m}{3} \right) q^{nm} \\ &= \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{69}{d} \right) \left(\frac{n/d}{3} \right) \right) q^n, \end{aligned}$$

and so the coefficient of q^n in $L_2(q)$ is given by

$$B(n) := [q^n] \sum_{n=1}^{\infty} \left(\frac{69}{n} \right) \frac{q^n(1-q^n)}{1-q^{3n}} = \sum_{d|n} \left(\frac{69}{d} \right) \left(\frac{n/d}{3} \right).$$

It is easy to check that for a prime p

$$(7.12) \quad A(p^\alpha) = \begin{cases} 1 & p = 23, \\ 1 + \alpha & \left(\frac{-23}{p} \right) = 1, \\ \frac{(-1)^\alpha + 1}{2} & \left(\frac{-23}{p} \right) = -1, \end{cases}$$

and

$$(7.13) \quad B(p^\alpha) = \begin{cases} 0 & p = 3, \alpha \neq 0 \\ (-1)^\alpha & p = 23, \\ 1 + \alpha & \left(\frac{-23}{p}\right) = 1, \text{ and } \left(\frac{p}{3}\right) = 1, \\ (-1)^\alpha(1 + \alpha) & \left(\frac{-23}{p}\right) = 1, \text{ and } \left(\frac{p}{3}\right) = -1, \\ \frac{(-1)^\alpha + 1}{2} & \left(\frac{-23}{p}\right) = -1. \end{cases}$$

Since $A(n)$ and $B(n)$ are multiplicative we can use (7.12) and (7.13) along with (7.10) and (7.11) to give formulas for the number of representations of an integer by a given genus of discriminant -207 .

Theorem 7.1. *Let the prime factorization of n be*

$$n = 3^a \cdot 23^b \prod_{i=1}^r p_i^{v_i} \prod_{j=1}^s q_j^{w_j},$$

where $p_i \neq 3$ and $\left(\frac{-23}{p_i}\right) = 1$ and $\left(\frac{-23}{q_j}\right) = -1$. Let

$$\Lambda(n) := \prod_{i=1}^r (1 + v_i) \prod_{j=1}^s \frac{1 + (-1)^{w_j}}{2}.$$

We have

$$(7.14) \quad (1, 1, 52; n) + 2(4, 1, 13; n) = \begin{cases} (1 + (-1)^{b+t})\Lambda(n) & a = 0, \\ 0 & a = 1, \\ 2\Lambda(n) & a \geq 2, \end{cases}$$

and

$$(7.15) \quad (8, 7, 8; n) + 2(2, 1, 26; n) = \begin{cases} (1 - (-1)^{b+t})\Lambda(n) & a = 0, \\ 0 & a = 1, \\ 2\Lambda(n) & a \geq 2, \end{cases}$$

where t is the number of prime factors p of n , counting multiplicity, with $\left(\frac{-23}{p}\right) = 1$ and $\left(\frac{p}{3}\right) = -1$.

Theorem 7.1 gives the total number of representations by all forms of a given genus of discriminant -207 . To find $(a, b, c; n)$ for any particular form of discriminant -207 , one can employ the techniques of [1].

Let $A = (2, 1, 26)$ and $A(q)$ the associated theta series. Recall $\text{CL}(-207) \cong \mathbb{Z}_6$, and A is a generator of this group. Theorem 7.1 gives representation formulas for

$$(7.16) \quad I(q) + 2A^2(q),$$

$$(7.17) \quad A^3(q) + 2A(q),$$

where I is the principal form, and A^k corresponds to Gaussian composition k times. The techniques of [1] allows us to find representation formulas for

$$(7.18) \quad I(q) - A^2(q),$$

$$(7.19) \quad A(q) - A^3(q),$$

by using the fact that

$$(7.20) \quad M(q) := \frac{I(q) - A^2(q) + [A(q) - A^3(q)]}{2},$$

is an eigenform for all Hecke operators and also employing congruences to separate $I(q) - A^2(q)$ and $A(q) - A^3(q)$. We note that one can use the formulas of Hecke [6, p.794] to show $M(q)$ is an eigenform for all Hecke operators. A concise formula for the action of the Hecke operators on the theta series associated to a binary quadratic form is given by [1, (1.18)]. It is interesting to note that

$L_2(q) = \frac{I(q)+2A^2(q)-(A^3(q)+2A(q))}{2}$ is an example of a Lambert series which is an eigenform for all Hecke operators.

[1] discusses the example $\text{CL}(-135) \cong \mathbb{Z}_6 \cong \langle A \rangle$ which is very similar to our example except that for $\text{CL}(-135)$ both $\frac{I(q)-A^2(q) \pm [A(q)-A^3(q)]}{2}$ are eigenforms for all Hecke operators. In our example the combination $\frac{I(q)-A^2(q)-[A(q)-A^3(q)]}{2}$ is not an eigenform for all Hecke operators and so congruences must be employed to derive (7.18). We do not give explicit representation formulas for (7.18) since the derivation process is similar to the example for $\Delta = -135$ in [1].

Another approach to proving Theorem 7.1 is to employ the general formula [7, Theorem 8.1, pg. 289] proven by Huard, Kaplan, and Williams. This formula gives the total number of representations of an integer n by all the forms in a genus of discriminant $d < 0$. Section 8 of [9] discusses representations of n by an individual form. We conclude this paper by deriving Theorem 7.1 from Theorem 8.1 in [7].

In the case that $9 \mid n > 0$, Theorem 8.1 of [7] gives

$$(1, 1, 52; n) + 2(4, 1, 13; n) = (8, 7, 8; n) + 2(2, 1, 26; n) = 2 \sum_{\mu \mid \frac{n}{9}} \left(\frac{-23}{\mu} \right),$$

which is consistent with Theorem 7.1. When $3 \mid n$ and $9 \nmid n$ Theorem 8.1 of [7] gives

$$(1, 1, 52; n) + 2(4, 1, 13; n) = (8, 7, 8; n) + 2(2, 1, 26; n) = 0.$$

The last case to consider is when $3 \nmid n$. Using the notation of [7], Theorem 8.1 of [7] gives the total number of representations by the forms of genus G to be

$$R_G(n, -207) = \frac{1}{2} \sum_{d_1 \in \{1, -3, -23, 69\}} \gamma_{d_1}(G) S(n, d_1, \frac{-207}{d_1}),$$

where

$$S(n, d_1, \frac{-207}{d_1}) = \sum_{\mu\nu=n} \left(\frac{d_1}{\mu} \right) \left(\frac{-207/d_1}{\nu} \right),$$

and $\gamma_{d_1}(G) = \left(\frac{d_1}{g} \right)$ with g a positive integer coprime to d_1 and represented by the genus G . Simplifying yields

$$(1, 1, 52; n) + 2(4, 1, 13; n) = \sum_{\mu \mid n} \left(\frac{-23}{\mu} \right) + \sum_{\mu\nu=n} \left(\frac{-3}{\mu} \right) \left(\frac{69}{\nu} \right),$$

and

$$(8, 7, 8; n) + 2(2, 1, 26; n) = \sum_{\mu \mid n} \left(\frac{-23}{\mu} \right) - \sum_{\mu\nu=n} \left(\frac{-3}{\mu} \right) \left(\frac{69}{\nu} \right),$$

which is consistent with Theorem 7.1 in this case.

8. ACKNOWLEDGMENTS

The author would like to thank Alexander Berkovich for valuable discussions. Also the author is grateful to the anonymous referee for helpful comments and for bringing references [7] and [9] to his attention.

REFERENCES

- [1] A. Berkovich and F. Patane, *Binary quadratic forms and the Fourier coefficients of certain weight 1 eta-quotients*, J. Number Theory **135**, (2014), 185–220.
- [2] D. Buell, *Binary Quadratic Forms: Classical Theory and Modern Computations*, Springer-Verlag, New York, 1990.
- [3] D. A. Cox, *Primes of the Form $x^2 + ny^2$: Fermat Class Field Theory and Complex Multiplication*, John Wiley & Sons, New York, 1989.
- [4] L. E. Dickson, *Introduction to the Theory of Numbers*, Dover, New York, 1957.
- [5] C. F. Gauss, *Disquisitiones Arithmeticae*, Yale University Press, 1965.

- [6] E. Hecke, *Mathematische Werke*, Vandenhoeck & Ruprecht, Göttingen, 1970.
- [7] J. G. Huard, P. Kaplan, and K. S. Williams, *The Chowla-Selberg formula for genera*, Acta Arithmetica **73**, (1995), 271–301.
- [8] F. Patane, *An identity connecting theta series associated with binary quadratic forms of discriminant Δ and $\Delta(\text{prime})^2$* , J. Number Theory **156**, (2015), 290–316.
- [9] Z. H. Sun, K. S. Williams, *On the number of representations of n by $ax^2 + bxy + cy^2$* , Acta Arithmetica **122**, (2006), 101–171.

DEPARTMENT OF MATHEMATICS, SAMFORD UNIVERSITY, 800 LAKESHORE DRIVE, BIRMINGHAM AL 35209, USA
E-mail address: fpatane@samford.edu